

GREEN'S FORMULA AND VARIATIONAL INEQUATIONS

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ABSTRACT. In this paper, I give an extension to the banachic - non linear- case of the work of B. Hanouzet and J.L. Joly which is quoted below in the bibliography and which was elaborated in the hilbertian-linear- case.

1. INTRODUCTION

Let $\mathcal{L}, \mathcal{B}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{T}$ be five reflexive Banach spaces such that :

(*) \mathcal{B}_0 (resp. \mathcal{B}_1) is a dense vectorial subspace of \mathcal{L} and j_0 (resp. j_1) is continuous injection from \mathcal{B}_0 (resp. \mathcal{B}_1) into \mathcal{L} .

(**) \mathcal{B}_0 is a vectorial subspace of \mathcal{B}_1 , embedded with the norm of \mathcal{B}_1 , which is closed in \mathcal{B}_1 .

We shall denote by j the (continuous) injection of \mathcal{B}_0 into \mathcal{B}_1 .

(***) Γ is the (closed) image of \mathcal{B}_1 in \mathcal{T} by a linear mapping γ_0 such that :

$$\text{Im } j = \ker \gamma_0.$$

Let us denote by $\mathcal{L}', \mathcal{B}', \mathcal{B}'_0, \mathcal{B}'_1, \mathcal{T}'$ the topological duals of $\mathcal{L}, \mathcal{B}, \mathcal{B}_0, \mathcal{B}_1, \mathcal{T}$ respectively.

Then, ${}^t j_0$ (resp. ${}^t j_1$) is a (linear and continuous) injective mapping from \mathcal{L}' into \mathcal{B}'_0 (resp. \mathcal{B}'_1) with a dense image for the strong topology.

As : $j_0 = j_1 j$, we have : ${}^t j_0 = {}^t j {}^t j_1$. We set : $\rho = {}^t j$.

As $\text{Im}(j)$ is closed, $\text{Im}({}^t j)$ is also closed.

${}^t \gamma_0$ is a (linear and continuous) injective mapping from \mathcal{T}' into \mathcal{B}'_1 (embedded with its strong topology).

Moreover, $\text{Im}({}^t \gamma_0)$ is closed (as $\text{Im}(\gamma_0)$ is closed) and we have :

$$\text{Im}({}^t \gamma_0) = \ker({}^t j).$$

So, we have the following scheme :

$$\begin{array}{ccccccc} & & \mathcal{T}' & \longrightarrow & \mathcal{T} & & \\ & & \downarrow & & \uparrow & & \\ \mathcal{L}' & \longrightarrow & \mathcal{B}'_1 & \longrightarrow & \mathcal{B}_1 & \longrightarrow & \mathcal{L} \\ & \searrow & \downarrow & & \uparrow & \nearrow & \\ & & \mathcal{B}'_0 & \longrightarrow & \mathcal{B}_0 & & \end{array}$$

Below, we denote by $\|\cdot\|_{\mathcal{L}}$ (resp. $\|\cdot\|_1, \|\cdot\|_0, \|\cdot\|_{\mathcal{T}}$) the norm of \mathcal{L} (resp. $\mathcal{B}_1, \mathcal{B}_0, \mathcal{T}$) and by $\langle \cdot, \cdot \rangle_{\mathcal{L}', \mathcal{L}}$ (resp. $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_0, \langle \cdot, \cdot \rangle_{\mathcal{T}', \mathcal{T}}$) the duality bracket between \mathcal{L} and \mathcal{L}' (resp. \mathcal{B}_1 and $\mathcal{B}'_1, \mathcal{B}_0$ and $\mathcal{B}'_0, \mathcal{T}$ and

\mathcal{T}').

So, we have the following scheme :

$$\begin{array}{ccccc} \mathcal{L}' & \longrightarrow & {}^t j_1 \mathcal{L}' & \longrightarrow & \mathcal{B}'_1 \\ & \searrow & \uparrow & & \downarrow \\ & & {}^t j_1 \mathcal{L}' & \longrightarrow & \mathcal{B}'_0 \end{array}$$

The restriction of ρ to ${}^t j_1 \mathcal{L}'$ is injective and surjective as ${}^t j_0 = \rho \circ {}^t j_1$ and ${}^t j_0$ (${}^t j_1$) are injective.

The restriction of ρ to ${}^t j_0 \mathcal{L}'$ is bijective. Its inverse denoted by $\bar{\omega}$ is bijective.

(We don't know, however, if $\bar{\omega}$ is continuous).

Example 1 :

$\forall u, v \in H^1(0, 1)$, let : $a(u, v) = \int_0^1 (u'v' + uv) dt = \langle Au, v \rangle_1$.

Then : $\forall u, v \in H^1(0, 1)$, $a(u, v) = \langle Lu, v \rangle_0$, thus : $L = \rho A$, and :

$\forall u, v \in C^\infty(0, 1)$, $\int_0^1 (u'v' + uv) dt = \int_0^1 (-u'' + u) \cdot v dt + u'(1)v(1) - u'(0)v(0)$.

i.e. $\langle Au, v \rangle_1 = \langle \pi Lu, v \rangle_1 = \langle \gamma_0 u, \gamma_0 v \rangle_{\mathcal{T}', \mathcal{T}}$.

If, $v \in \mathcal{D}(0, 1)$, then : $\langle \pi Lu, v \rangle_1 = \int_0^1 (-u'' + u) v dt = \left\langle (-u'' + u), v \right\rangle_{\mathcal{D}'(0,1), \mathcal{D}(0,1)}$.

Example 2 :

Let $p \in \mathbb{N}^*$.

$\forall u, v \in \mathcal{B}_1$, $h(u, v) = \int_0^1 (|u'|^{p-1} \text{sign}(u') \cdot v') dt = \langle Gu, v \rangle_1$ and :

$\forall u \in \mathcal{B}_1, \forall v \in \mathcal{B}_0$, $h(u, v) = \int_0^1 (|u'|^{p-1} \text{sign}(u') \cdot v') dt = \langle \rho Gu, v \rangle_0$

where :

ρGu is the restriction of Gu to \mathcal{B}'_0 .

Formally, we can write :

$$\langle Gu, v \rangle_1 = \langle \pi \rho Gu, v \rangle_1 + \langle \gamma_G u, \gamma_0 v \rangle_{\mathcal{T}', \mathcal{T}}$$

We do not need to precise, here, the definition of \mathcal{B}_0 and \mathcal{B}_1 .

However, if $p > 1$, $\mathcal{B}_1 = \{u \in \mathcal{D}'(0, 1) ; u, u' \in L^p(0, 1) \text{ and } u(0) = 0\}$

and $\mathcal{B}_0 = \{v \in \mathcal{B}_1 ; \gamma_1 v = 0\}$ with : $\forall v \in C^1(0, 1)$, $\gamma_1 v = v'(1)$

we deduce that, if u is sufficiently regular, $\gamma_G u = (|u'|^{p-1} \text{sign}(u'))(1)$.

(cf. " Quelques méthodes de résolution de problèmes aux limites non linéaires", de J.L. Lions, p.25).

2. GREEN'S FORMULA.

We recall that $\bar{\omega}$ is a linear mapping from ${}^t j_0 \mathcal{L}'$ into \mathcal{B}'_1 such that :

$$\forall l' \in \mathcal{L}', \quad \bar{\omega} {}^t j_0 l' = {}^t j_1 l'.$$

Let G be an operator (which is linear or not) continuous from \mathcal{B}_1 into \mathcal{B}'_1 . We set :

$$\mathcal{B}_G = \{u \in \mathcal{B}_1 ; \rho G u \in {}^t j_0 \mathcal{L}'\}$$

Then :

$$\forall u \in \mathcal{B}_G, \exists l' \in \mathcal{L}' \text{ such that : } {}^t j (G u - {}^t j_1 l') = \theta_{\mathcal{B}'_0} \\ \text{where } \theta_{\mathcal{B}'_0} \text{ is the neutral element of } \mathcal{B}'_0$$

Thus, we deduce that :

$$G u - {}^t j_0 l' = G u - \overline{\omega} {}^t j_0 l' \in \ker {}^t j = \text{Im } \gamma_0$$

So :

$$G u - \overline{\omega} {}^t j_0 l' = G u - \overline{\omega} \rho G u = {}^t \gamma_0 z', \text{ with } z' \in \mathcal{T}'$$

We shall set, in the following :

$$\forall u \in \mathcal{B}_G, \gamma_G u = {}^t \gamma_0^{-1} (I'_1 - \overline{\omega} \rho) G u \in \mathcal{T}'$$

where I'_1 is the identity of \mathcal{B}'_1 .

Remarks :

(*) γ_G is a mapping from \mathcal{B}_G into \mathcal{T}' which is not necessarily linear, but ${}^t \gamma_0^{-1} (I'_1 - \overline{\omega} \rho)$ is a linear mapping which is definite on $G(\mathcal{B}_G)$.

(**) $(u \in \mathcal{B}_G) \iff \chi_1^{-1} J_1^* G u \in \mathcal{B}_{J_1}$.

Then :

$$\forall u \in \mathcal{B}_G, \forall y \in \mathcal{B}_1, \langle G u, y \rangle_1 = \langle \overline{\omega} \rho G u, y \rangle_1 + \langle \gamma_G u, \gamma_0 y \rangle_{\mathcal{T}', \mathcal{T}}$$

Thus, we have the following scheme :

$$\begin{array}{ccccccc} & & \mathcal{T}' & \longrightarrow & \mathcal{T} & & \\ & & \downarrow & & \uparrow & & \\ E' & \longrightarrow & \mathcal{L}' & \longrightarrow & \mathcal{B}'_1 & \longrightarrow & \mathcal{B}_1 \longrightarrow \mathcal{L} \longrightarrow E \\ & & \searrow & & \downarrow & & \uparrow \nearrow \\ & & & & \mathcal{B}'_0 & \longrightarrow & \mathcal{B}_0 \end{array}$$

So :

$$l_0 = l_1 \circ j, \quad {}^t l_0 = \rho \circ {}^t l_1, \quad l_1 = k \circ j_1, \quad l_0 = k \circ j_0 \\ \text{with } \forall e' \in E', \quad \overline{\omega} {}^t l_0 e' = {}^t l_1 e'$$

Example :

$$\mathcal{B}_1 = W^{1,p}(\Omega), \quad \mathcal{B}_0 = W_0^{1,p}(\Omega), \quad \mathcal{L} = L^p(\Omega), \quad E = \mathcal{D}'(\Omega)$$

Now, we suppose that j_1 (resp. j_0, k) is a weakly dense injection (and weakly continuous) from \mathcal{B}_1 into \mathcal{L} (resp. \mathcal{B}_0 into \mathcal{L} , \mathcal{L} into E).

Then : ${}^t j_1$ (resp. ${}^t j_0, {}^t k$) is injective and its image is weakly (strongly) dense in \mathcal{B}'_1 (resp. $\mathcal{B}'_0, \mathcal{L}'$) when those vector spaces are Banach spaces (reflexive or not).

Let us set :

$$\mathcal{B}_{G,E} = \{u \in \mathcal{B}_1; \rho G u \in {}^t l_0 E'\} \subset \mathcal{B}_G$$

So :

$$\forall u \in \mathcal{B}_{G,E}, \exists e' \in E', \text{ such that : } {}^t j (G u - {}^t l_1 l') = \theta_{\mathcal{B}'_0} \\ \implies G u - {}^t l_1 e' = G u - \overline{\omega}_E {}^t l_0 e' \in \ker {}^t j = \text{Im } \gamma_0 \\ \implies G u - \overline{\omega} {}^t l_0 e' = G u - \overline{\omega}_E \rho G u = {}^t \gamma_0 z', \quad z' \in \mathcal{T}'$$

Now, we set :

$$\forall u \in \mathcal{B}_{G,E}, \quad \gamma_G u = {}^t\gamma_0^{-1} (I'_1 - \overline{\omega}_E \rho) G u \in \mathcal{T}'$$

Remark :

$$\forall e' \in E', \quad \overline{\omega}_E {}^tj_0 {}^tk e' = {}^tj_1 {}^tk e' = \overline{\omega} {}^tj_0 {}^tk e'$$

Thus : $\overline{\omega}_E$ is the restriction of $\overline{\omega}$ to ${}^tj_0 {}^tk E'$ which is contained in ${}^tj_0 \mathcal{L}'$, and :

$$\forall u \in \mathcal{B}_{G,E}, \quad \forall y \in \mathcal{B}_1, \quad \langle Gu, y \rangle_1 = \langle \overline{\omega}_E \rho G u, y \rangle_1 + \langle \gamma_{G,E} u, \gamma_0 y \rangle_{\mathcal{T}', \mathcal{T}}$$

where $\gamma_{G,E}$ is the restriction of γ_G to $\mathcal{B}_{G,E} \subset \mathcal{B}_G$.

3. AN ORDER ON \mathcal{L} AND ITS INDUCED ORDER ON \mathcal{B}_1 (RESP. \mathcal{B}_0).

Let C^+ be a closed convex cone in \mathcal{L} with $\theta_{\mathcal{L}}$ as its vertex.

Then :

$$\mathcal{L} = C^+ - C^+ \quad \text{and} \quad \theta_{\mathcal{L}} = C^+ \cap (-C^+)$$

Let :

$$C_m^+ = \{u \in \mathcal{B}_m ; j_m u \in C^+\} \quad \text{for } m = 0, 1$$

We have :

$$\mathcal{B}_m = C_m^+ - C_m^+ \quad \text{and} \quad \theta_{\mathcal{B}_m} = C_m^+ \cap (-C_m^+) \quad \text{for } m = 0, 1$$

Hypotheses :

We suppose that :

(*) for $m = 0, 1$, $j_m C_m^+$ is dense in C^+ (for the topology of \mathcal{L}).

(**) \mathcal{B}_m is nested for the order induced by that of \mathcal{L} .

If $u \in [C^+]$ (resp. C_1^+, C_0^+), we shall write : $u \geq 0$ (and $u \leq 0$ if $u \in (-C^+) = C^-$ (resp. $-C_m^+ = C_m^-$) for $m = 0, 1$)

We shall denote by D^+ (resp. D_1^+, D_0^+) the polar cone of C^+ (resp. C_1^+ , C_0^+) for the duality between \mathcal{L} and \mathcal{L}' (resp. \mathcal{B}_1 and \mathcal{B}_1' , \mathcal{B}_0 and \mathcal{B}_0'). So :

$$D^+ = \{x^* \in \mathcal{L}' ; \forall x \in C^+, \langle x^*, x \rangle \geq 0\}$$

D^+ (resp. D_m^+ for $m = 0, 1$) is a weakly (and strongly) closed cone as \mathcal{B} (resp. \mathcal{B}_m for $m = 0, 1$) is reflexive.

If $u^* \in D^+$ (resp. D_m^+ for $m = 0, 1$), we shall write $u^* \geq 0$ and $u^* \leq 0$ if $u \in (-D^+) = D^-$ (resp. $(-D_m^+) = D_m^-$) for $m = 0, 1$.

Proposition 1. D_m^+ is the adherence of ${}^tj_m D^+$ in \mathcal{B}_m' for $m = 0, 1$.

Proof. First, we remark that : ${}^tj_m D^+ \subset D_m^+$ (we have : ${}^tj_0 D^+ = \rho {}^tj_1 D^+$).

Indeed : $\forall l' \in D^+, \forall u \in C_m^+, \langle {}^tj_m l', u \rangle_m = \langle l', j_m u \rangle_{\mathcal{L}', \mathcal{L}} \geq 0$.

Let us suppose that : ${}^tj_m D^+ \subsetneq D_m^+$ and let : $u' \in D_m^+ \setminus \overline{{}^tj_m D^+}$.

As $\overline{{}^tj_m D^+}$ is a closed convex cone (with $\theta_{\mathcal{B}_m'}$ as its vertex)

and as \mathcal{B}_m is reflexive, there exists $u \in \mathcal{B}_m$ such that :

$$\forall l' \in D^+, \quad \langle {}^tj_m l', u \rangle_m \geq 0 \quad \text{and} \quad \langle u', u \rangle_m < 0$$

Thus, we have :

$$\forall l' \in D^+ , \quad \langle l', j_m u \rangle \geq 0 \quad \text{and} : \quad \langle u', u \rangle_m < 0$$

So :

$$(j_m u \geq 0 , \quad u \geq 0 \quad \text{and} : \quad \langle u', u \rangle_m < 0) \quad \text{what is impossible.}$$

□

Lemma 2. $\bar{\omega}$ is an homomorphism for the order structures which are definite on ${}^t j_0 \mathcal{L}'$ and ${}^t j_1 \mathcal{L}'$ respectively by ${}^t j_0 D^+$ and ${}^t j_1 D^+$.

Moreover : $\bar{\omega}({}^t j_0 D^+) = {}^t j_1 D^+$.

Proof. $(l' \in \mathcal{L}' , {}^t j_m l' \geq 0) \iff (l' \in \mathcal{L}' , l' \geq 0)$ for $m = 0, 1$ because
 $({}^t j_m l' \geq 0) \iff (\forall v \in C_m^+ , \langle {}^t j_m l', v \rangle_m = \langle l', j_m v \rangle_m \geq 0)$
 $\iff (\forall u \in C^+ , \langle l', u \rangle \geq 0)$ (because ${}^t j_m C^+$ is dense in C^+).
 Thus : $({}^t j_0 l' \geq 0) \implies (\bar{\omega} {}^t j_0 l' = {}^t j_1 l' \geq 0)$.

□

Lemma 3. $\forall u \in C_1^+ , \forall v' \in \rho D_1^+ , \forall w' \in (\rho)^{-1} v' ,$

$$\text{Sup} \{ \langle v', v \rangle_0 ; 0 \leq v \leq u , v \in \mathcal{B}_0 \} \leq \langle w', u \rangle_1 .$$

Proof.

$$\forall v \in \mathcal{B}_0 , 0 \leq v \leq u , \langle v', v \rangle_0 = \langle \rho w', v \rangle_0 = \langle w', jv \rangle_1 \leq \langle w', u \rangle_1$$

□

Lemma 4. $\forall u \in C_1^+ , \forall v' \in {}^t j_0 D^+ = \rho {}^t j_1 D_1^+ , \langle \bar{\omega} v', u \rangle_1 = \text{Sup} \{ \langle v', v \rangle_0 ; 0 \leq v \leq u \}$

Proof. Let $v \in C_0^+$ such that : $u - jv \in C_1^+$. Then :

$$\langle \bar{\omega} v', u - jv \rangle_1 = \langle {}^t j_1 l', u - jv \rangle_1 = \langle l', j_1 (u - jv) \rangle_{\mathcal{L}', \mathcal{L}} \geq 0 \quad (\text{where } l' \in D^+)$$

Thus :

$$\langle \bar{\omega} v', u \rangle_1 \geq \langle v', v \rangle_0 .$$

On the other hand, we know that there exists a sequence $(v_m ; m \in \mathbb{N})$ contained in C_0^+ such that :

$$\forall p \in \mathbb{N} , u - jv_p \in C_1^+ \quad \text{and} \quad \lim_{p \rightarrow \infty} \|j_0 v_p - j_1 u\|_{\mathcal{L}} = 0$$

We deduce that :

$$\forall p \in \mathbb{N} , \langle \bar{\omega} v', u \rangle_1 \geq \langle v', v_p \rangle_0$$

and :

$$\begin{aligned} \lim_{p \rightarrow \infty} \langle v', v_p \rangle_0 &= \lim_{p \rightarrow \infty} \langle {}^t j_0 l', v_p \rangle_0 = \lim_{p \rightarrow \infty} \langle l', j_0 v_p \rangle_{\mathcal{L}', \mathcal{L}} \\ &= \langle l', j_1 u \rangle_{\mathcal{L}', \mathcal{L}} = \langle {}^t j_1 l', u \rangle_1 = \langle \bar{\omega} v', u \rangle_1 \end{aligned}$$

□

Proposition 5. $\forall u \in C_1^+ , \forall v' \in {}^t j_0 D^+ = \rho {}^t j_1 D_1^+ ,$

$$\langle \bar{\omega} v', u \rangle_1 = \text{Sup} \{ \langle v', v \rangle_0 ; 0 \leq v \leq u \} = \text{Inf} \{ \langle w', u \rangle_1 ; u' \in (\rho)^{-1} v' \}$$

Proof. deduced from lemmas 2 and 3 and from the fact that : $\bar{\omega} v' \in (\rho)^{-1} v'$

□

References.

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